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**Ruled minimal surfaces in the three-dimensional Heisenberg group. (English summary)**

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In the article under review, the authors endow the three-dimensional Heisenberg group with the standard Riemannian metric or with the standard Lorentzian metric.

For the first choice of metric, they classify in $\mathrm{Nil}^3$ minimal surfaces ruled by geodesics. Namely, they show that, up to isometry, the minimal surface is a part of a plane, a helicoid or a hyperbolic paraboloid. To attain this result they prove a lemma that states that if a ruled surface is minimal and if a ruling geodesic is not tangent to the fiber, then the ruled surface should be horizontally ruled. In addition, it is proved that all ruled minimal surfaces in $\mathrm{Nil}^3$ are either helicoids or limits of sequences of helicoids in the Hausdorff sense. Also, the classification result is compared to one that was obtained in [M. Bekkar and T. Sari, Rend. Sem. Mat. Univ. Politec. Torino **50** (1992), no. 3, 243–254 (1993); MR1249465 (94h:53009)]. It is explained why the condition that the geodesic rulings are straight lines in that reference is redundant.

Finally, it is argued that surfaces for which the mean curvature is zero with respect to both metrics under consideration are also parts of planes, helicoids or hyperbolic paraboloids.

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*Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.*